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Eigenvalue bounds of structures with uncertain-but-bounded parameters

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Abstract

Many analysis and design problems in engineering and science involve uncertainty to varying degrees. This paper is concerned with the structural vibration problem involving uncertain material or geometric parameters, specified as bounds on these parameters. This produces interval stiffness and mass matrices, and the problem is transformed into a generalized interval eigenvalue problem in interval mathematics. However tighter bounds on the eigenvalues may be obtained by using the formulation of the structural dynamic problem. Often the stiffness and mass matrices can be formed as a non-negative decomposition in the uncertain structural parameters. In this case the eigenvalue bounds may be obtained from the parameter vertex solutions. Even more efficiently, using interval extension from interval mathematics, the generalized interval eigenvalue problem may be divided into two generalized eigenvalue problems for real symmetric matrix pairs. The parameter vertex solution algorithm is compared with Deif's solution, the eigenvalue inclusion principle and the interval perturbation method in numerical examples.

1. Introduction

All structures are subject to uncertainties of one form or another. The response of engineering structures is affected by the uncertainties in the parameters and loading. The uncertainty may be

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specified in a number of ways, such as probablistic [1], convex [2] or fuzzy [3] descriptions. This paper concentrates on approximating the uncertain data using intervals, which may be classified within the convex set descriptions. Suppose a is a structural parameter vector such that

$$\underline{a} \leqslant a \leqslant \bar{a} \quad \text{or} \quad a \in [\underline{a}, \bar{a}] = a^I, \tag{1}$$

where $\underline{a}$ and $\bar{a}$ denote the lower and upper bounds of the vector a, and a^I represents the closed interval that contains a, and is called the interval parameter. The influence of these errors or uncertainties on the response of the structure, $R(a)$, is of great interest. This is equivalent to finding the lower and upper bounds, $\underline{R}$ and $\bar{R}$, such that

$$\underline{R} \leqslant R(a) \leqslant \bar{R} \quad \text{for } \underline{a} \leqslant a \leqslant \bar{a}. \tag{2}$$

$R^I = [\underline{R}, \bar{R}]$ is referred to as the interval response, and is

$$R^I = [\underline{R}, \bar{R}] = \{R(a)\colon a \in a^I = [\underline{a}, \bar{a}]\}. \tag{3}$$

As an illustration of a system with an interval parameter, consider the single degree of freedom spring–mass system shown in Fig. 1. The eigenvalue, which is the natural frequency squared, is

$$\lambda = \omega^2 = \frac{k}{m}. \tag{4}$$

Now let the spring stiffness k and the mass m be interval variables with intervals $k^I = [\underline{k}, \bar{k}]$ and $m^I = [\underline{m}, \bar{m}]$. The objective is to find the lower and upper bounds, $\underline{\lambda}$ and $\overline{\lambda}$, of the eigenvalue λ of the spring–mass system. By natural interval extension [4],

$$\lambda^I = [\underline{\lambda}, \bar{\lambda}] = \frac{[\underline{k}, \bar{k}]}{[\underline{m}, \bar{m}]} = \left[\frac{\underline{k}}{\bar{m}}, \frac{\bar{k}}{\underline{m}}\right]. \tag{5}$$

The eigenvalue bounds are then

$$\underline{\lambda} = \frac{\underline{k}}{\bar{m}}, \quad \bar{\lambda} = \frac{\bar{k}}{\underline{m}}. \tag{6}$$

This corresponds to the usual engineering understanding, that the lowest eigenvalue is obtained when the spring stiffness is at its lowest value, and the mass is at its highest value, and similar for the upper bound on the eigenvalues. Notice that in Eq. (5) the interval variables k^I and m^I occur only once, and hence, Eq. (5) yields the exact interval of the eigenvalue. An alternative

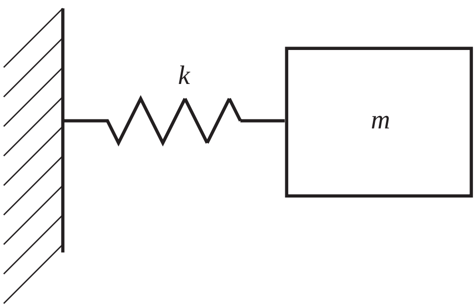

Fig. 1. The single degree of freedom spring–mass system.

interpretation of the interval eigenvalue is to take the midpoint, $\lambda^c = (\bar{\lambda} + \underline{\lambda})/2$ as an approximation to λ, and half of the width, $\Delta\lambda = (\bar{\lambda} - \underline{\lambda})/2$, as the uncertainty. Thus, the computation of an interval eigenvalue containing an exact eigenvalue provides both an approximation to the exact eigenvalue and error bounds on the approximate eigenvalue.

To find the range of the structural responses due to random structural parameters, the theory of probabilitity may be used for cases capable of exact solution or otherwise Monte Carlo simulation may be used. Such a calculation demands knowledge of the probability density functions of every structural parameter, including their joint densities. Often there is insufficient probabilistic information, and the set-theoretic approach independently pioneered by Schweppe [5] in control theory and Drenick [6] in the response of structures to earthquakes. The convex modeling of uncertainty was independently developed and applied by Ben-Haim and Elishakoff [2] in applied mechanics. It should be stressed that the non-probabilistic, set-theoretic representation of uncertainty (dubbed as unknown-but-bounded or uncertain-but-non-random model) in the parametric space is motivated by the lack of detailed probabilistic information on the possible distributions of the parameters. Non-probabilistic, set-theoretic modeling has been employed in a wide range of engineering applications [2,7,8].

In this study, a new method for solving the generalized interval eigenvalue problem is presented. Parameter uncertainties may cause significant changes in the natural frequencies or eigenvalues of structures, and in particular, they may cause the occurrence of mode localization which can be used as a means of passive control of vibrations. The interval eigenvalue problem has emerged in recent years as scientists and engineers have begun to realize its wide applicability. Rohn [9] studied the standard interval eigenvalue problem of a symmetric interval matrix and derived formulas for interval eigenvalues when the error matrix has rank one. Hallot and Bartlett [10] discovered that the spectrum of the eigenvalues of an interval matrix family depends on the spectrum of its extremes set. Hudak [11] investigated ways to relate this to the eigenvalues of a constant matrix under certain conditions. Based on the invariance properties of the characteristic vector entries, Deif [12] presented a method to compute interval eigenvalues for the standard interval eigenvalue problem. Qiu et al. [13] extended Deif's method to the generalized interval eigenvalue problem. The lack of an efficient criterion to judge the invariance properties of the signs of the components of the eigenvectors under the interval operations, before computing interval eigenvalues, appears to restrict the applications of Deif's approach. To overcome this limitation of Deif's method, Qiu et al. [14] developed a method to compute interval eigenvalues by assuming positive semi-definiteness of the error in the interval matrix pair. For small errors in the matrices, Qiu et al. [15] presented an interval perturbation method for the interval eigenvalue problem.

By considering the characteristics of an engineering structure using the non-negative decomposition of the mass and stiffness matrices, this paper proposes a highly efficient and widely applicable method for the generalized interval problem.

2. The generalized interval eigenvalue problem

Eigenvalue problems are commonly encountered in structural stability and vibration analysis. In the general case the eigenvalues and eigenvectors are complex. However, when external forces are conservative, and no damping is considered in the structural analysis, the eigenvalues are real and are related to the vibration frequencies. Although this paper only considers the case of real eigenvalues of an undamped structure, this covers a significant range of analyses of interest. Indeed very few finite element models consider damping. The vibration analysis of an undamped structure leads to the generalized eigenvalue problem of the form

$$\mathbf{Ku} = \lambda\mathbf{Mu}, \tag{7}$$

where $\mathbf{K} = (k_{ij}) \in \mathbb{R}^{n\times n}$ is the stiffness matrix, $\mathbf{M} = (m_{ij}) \in \mathbb{R}^{n\times n}$ is the mass matrix and $\mathbf{u} = (u_i)$ is the eigenvector or the mode shape. The notation $\mathbf{K} = (k_{ij})$ indicates that k_{ij} is the (i,j)th element of the matrix $\mathbf{K}$. For vibration problems λ is the eigenvalue, or the square of the natural frequency, ω. $\mathbf{K}$ is symmetric and positive semi-definite, and $\mathbf{M}$ is symmetric and positive definite. The eigenvectors are often normalized with respect to the mass matrix such that

$$\mathbf{u}^{\mathrm{T}}\mathbf{Mu} = 1. \tag{8}$$

The mass and stiffness matrices are functions of the structural parameters, such as physical properties and geometric variables, and thus

$$\mathbf{K} = \mathbf{K}(\mathbf{b}), \quad \mathbf{M} = \mathbf{M}(\mathbf{b}), \tag{9}$$

where $\mathbf{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}$ is the structural parameter vector.

Experiments have shown that the eigenvalues and eigenvectors vary because the physical properties and geometric variables of the structure can neither be measured nor manufactured exactly. Thus, the eigenvalues and eigenvectors are uncertain variables whose uncertain properties are determined by the uncertain structural parameters of the stiffness and mass matrices. Thus, consider the eigenvalue problem (7), where the structural parameter vector lies in an interval, as

$$\underline{\mathbf{b}} \leqslant \mathbf{b} \leqslant \bar{\mathbf{b}} \quad \text{or} \quad \underline{b}_i \leqslant b_i \leqslant \bar{b}_i, \quad i = 1, 2, \ldots, m, \tag{10}$$

where $\underline{\mathbf{b}} = (\underline{b}_i)$ and $\bar{\mathbf{b}} = (\bar{b}_i)$ are the lower and upper bound vectors respectively of the structural parameter vector $\mathbf{b}$. In terms of interval matrix notation in interval analysis [4,16] the inequality condition (10) may be written as

$$\mathbf{b} \in \mathbf{b}^I \quad \text{or} \quad b_i \in b_i^I, \quad i = 1, 2, \ldots, m, \tag{11}$$

where

$$\mathbf{b}^I = (b_i^I), \quad b_i^I = [\underline{b}_i, \bar{b}_i], \quad i = 1, 2, \ldots, m. \tag{12}$$

$\mathbf{b}^I$ is the interval structural parameter, and b_i^I, $i = 1, 2, \ldots, m$, are the components of the interval vector [14,17].

The bounds on the eigenvalue set and the eigenvector set, subject to the constraint conditions (10) or (11), are

$$\Gamma = \{\lambda \colon \lambda \in \mathbb{R},\ \mathbf{K}(\mathbf{b})\mathbf{u} = \lambda\mathbf{M}(\mathbf{b})\mathbf{u},\ \mathbf{b} \in \mathbf{b}^I\}. \tag{13}$$

In general, the sets defined by Eq. (13) are very complicated; they are not interval vectors and need not be convex. The objective, therefore, is to determine a closed interval for each eigenvalue, λ_i^I, such that

$$\Gamma \subset \lambda^I = [\underline{\lambda}, \bar{\lambda}] = (\lambda_i^I), \quad \lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i], \quad i = 1, 2, \ldots, n, \tag{14}$$

where

$$\underline{\lambda}_i = \min_{\mathbf{b}\in\mathbf{b}^I} \lambda_i(\mathbf{K}(\mathbf{b}), \mathbf{M}(\mathbf{b})) \quad \text{and} \quad \overline{\lambda}_i = \max_{\mathbf{b}\in\mathbf{b}^I} \lambda_i(\mathbf{K}(\mathbf{b}), \mathbf{M}(\mathbf{b})). \tag{15}$$

The ith eigenvalue may be obtained from Eq. (7) or from

$$\lambda_i(\mathbf{K}(\mathbf{b}), \mathbf{M}(\mathbf{b})) = \min_{\mathbf{\Phi}_i \subset \mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i \\ \mathbf{u}\neq\mathbf{0}}} \left\{ \frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}} \right\}, \tag{16}$$

where $\mathbf{\Phi}_i \subset \mathbb{R}^n$ is an arbitrary i-dimensional subspace [18–21]. This is an extended form of the Rayleigh Quotient, where the size of the subspace of interest increases in dimension for the higher eigenvalues. The discussion also far has concentrated on estimating the uncertainty in the eigenvalues. The eigenvectors will also vary, and be contained within an interval vector. However, eigenvectors are generally less sensitive to parameter changes than the eigenvalues, and will not be considered further in this paper.

3. The non-negative decomposition of the matrix pair

Often, for the structural problems in engineering, the global mass and stiffness matrices may be written as a linear function of the structural parameters $\mathbf{b}=(b_i)$, so that

$$\begin{aligned} \mathbf{M}(\mathbf{b}) &= \mathbf{M}_0 + \sum_{i=1}^{m} b_i\mathbf{M}_i = \mathbf{M}_0 + b_1\mathbf{M}_1 + b_2\mathbf{M}_2 + \cdots + b_m\mathbf{M}_m, \\ \mathbf{K}(\mathbf{b}) &= \mathbf{K}_0 + \sum_{i=1}^{m} b_i\mathbf{K}_i = \mathbf{K}_0 + b_1\mathbf{K}_1 + b_2\mathbf{K}_2 + \cdots + b_m\mathbf{K}_m, \end{aligned} \tag{17}$$

where $\mathbf{M}_i$ and $\mathbf{K}_i$ are positive semi-definite and the parameters b_i are positive. This decomposition is called the *non-negative decomposition of a matrix*. Such decompositions arise naturally in a practical engineering context. For example, in structural finite element analysis, $\mathbf{M}_i$ and $\mathbf{K}_i$ may be taken as the element mass and stiffness matrices (or possibly substructure matrices) corresponding to the structural parameter b_i.

To further explain the *non-negative decomposition of a matrix*, consider a beam clamped at $x=0$ and free at $x = L_1 + L_2$, as shown in Fig. 2. Dividing the beam into two elements of lengths L_1

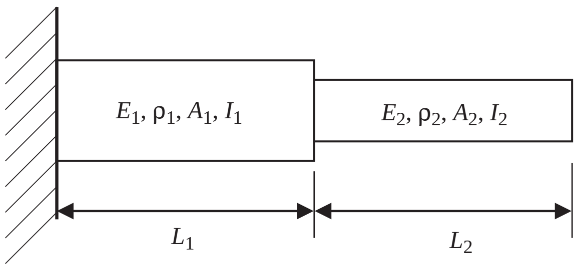

Fig. 2. The simple stepped beam.

and L_2, the element mass and stiffness matrices are

$$\rho_1 A_1 \mathbf{M}_1 = \frac{\rho_1 A_1 L_1}{420}\begin{bmatrix} 156 & -22L_1 & 0 & 0 \\ -22L_1 & 4L_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\rho_2 A_2 \mathbf{M}_2 = \frac{\rho_2 A_2 L_2}{420}\begin{bmatrix} 156 & 22L_2 & 54 & -13L_2 \\ 22L_2 & 4L_2^2 & 13L_2 & -3L_2^2 \\ 54 & 13L_2 & 156 & -22L_2 \\ -13L_2 & -3L_2^2 & -22L_2 & 4L_2^2 \end{bmatrix} \tag{18}$$

and

$$E_1 I_1 \mathbf{K}_3 = \frac{E_1 I_1}{L_1^3}\begin{bmatrix} 12 & -6L_1 & 0 & 0 \\ -6L_1 & 4L_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 I_2 \mathbf{K}_4 = \frac{E_2 I_2}{L_2^3}\begin{bmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4L_2^2 & -6L_2 & 2L_2^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2L_2^2 & -6L_2 & 4L_2^2 \end{bmatrix}. \tag{19}$$

The global mass and stiffness matrices are obtained by summing the element contributions, and are clearly of the form given in Eq. (17), with $\mathbf{b} = (\rho_1 A_1, \rho_2 A_2, E_1 I_1, E_2 I_2)^{\mathrm{T}}$, $\mathbf{M}_3 = \mathbf{M}_4 = 0$, $\mathbf{K}_1 = \mathbf{K}_2 = 0$, and the definitions of $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{K}_3$, $\mathbf{K}_4$ are obvious from Eqs. (18) and (19). Notice that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{K}_3$, $\mathbf{K}_4$ are all positive semi-definite.

Clearly, from Eq. (17), the elements m_{ij} and k_{ij} of the mass and stiffness matrices, $\mathbf{M}$ and $\mathbf{K}$, are functions of the structural parameters, $\mathbf{b}$, and by natural extension [4],

$$\mathbf{M}^I = [\underline{\mathbf{M}}, \bar{\mathbf{M}}] = \mathbf{M}_0 + \sum_{i=1}^{m} b_i^I \mathbf{M}_i = \mathbf{M}_0 + b_1^I \mathbf{M}_1 + b_2^I \mathbf{M}_2 + \cdots + b_m^I \mathbf{M}_m,$$

$$\mathbf{K}^I = [\underline{\mathbf{K}}, \bar{\mathbf{K}}] = \mathbf{K}_0 + \sum_{i=1}^{m} b_i^I \mathbf{K}_i = \mathbf{K}_0 + b_1^I \mathbf{K}_1 + b_2^I \mathbf{K}_2 + \cdots + b_m^I \mathbf{K}_m, \tag{20}$$

where $b_i^I = [\underline{b}_i, \bar{b}_i], i = 1, 2, \ldots, m$, are the uncertain-but-non-random parameters. These matrices may be defined element-wise. However it is clear, for example, that neither $\underline{\mathbf{K}}$ nor $\bar{\mathbf{K}}$ will necessarily be positive semi-definite. The structure of the mass and stiffness matrices given in Eq. (17) has been lost, and this is one reason for the conservative estimation of eigenvalue bounds.

4. The parameter vertex solution theorem

Before introducing the theorem that is the subject of this section, some notation is required. Suppose that the parameter interval vector is given by $\mathbf{b}^I$. Then the set of boundary vectors (sometimes called the extreme point vectors or the vertex vectors) of the parameter interval vector $\mathbf{b}^I$ is

$$\hat{\mathbf{b}} = \{\mathbf{b}\colon \mathbf{b} \in \mathbf{b}^I,\ \mathbf{b} = (b_i),\ \text{and}\ \hat{b}_i = \bar{b}_i\ \text{or}\ \hat{b}_i = \underline{b}_i, i = 1, 2, \ldots, m\}. \tag{21}$$

Hence the set of boundary vectors are the extreme vectors of the parameter interval vector $\mathbf{b}^I$ and contains 2^m elements, where m is the length of the vector $\mathbf{b}$. In the case of a parameter vector of length 2, the parameter interval vector is represented by a rectangle in parameter space, and the set of boundary vectors are the corners of the rectangle.

Parameter Solution Vertex Theorem. *Suppose that a non-negative decomposition of the mass and stiffnesses matrices exists, given by Eq.* (17) *and where the structural parameter vector* $\mathbf{b}$ *varys inside an interval parameter vector,* $\mathbf{b} \in \mathbf{b}^I$. *Then the ith eigenvalue,* λ_i, *is contained in an interval,* $\lambda_i \in \lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i]$, *where the lower and upper bounds of the eigenvalues are,*

$$\underline{\lambda}_i = \min\{\lambda_i(\mathbf{M}(\mathbf{b}), \mathbf{K}(\mathbf{b}))\colon \mathbf{b} \in \hat{\mathbf{b}}\}, \quad \bar{\lambda}_i = \max\{\lambda_i(\mathbf{M}(\mathbf{b}), \mathbf{K}(\mathbf{b}))\colon \mathbf{b} \in \hat{\mathbf{b}}\}. \tag{22}$$

Proof. The extremum values of λ_i, denoted $\lambda_{i\text{ext}}$, are

$$\begin{aligned}\lambda_{i\text{ext}} &= \underset{\mathbf{b}\in\mathbf{b}^I}{\text{extremum}}\{\lambda_i\} = \underset{\mathbf{b}\in\mathbf{b}^I}{\text{extremum}}\ \min_{\boldsymbol{\Phi}_i\subset\mathbb{R}^n}\ \max_{\substack{\mathbf{u}\in\boldsymbol{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}}\left\{\frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}}\right\}\\ &= \min_{\boldsymbol{\Phi}_i\subset\mathbb{R}^n}\ \max_{\substack{\mathbf{u}\in\boldsymbol{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}}\ \underset{\mathbf{b}\in\mathbf{b}^I}{\text{extremum}}\left\{\frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}}\right\}, \quad i = 1, 2, \ldots, n.\end{aligned} \tag{23}$$

To compute the extreme value of the quotient in Eq. (23) requires the condition that the mass and stiffness matrices have non-negative decompositions and that the elements of the parameter vector $\mathbf{b}$ are positive. The non-negative decomposition is of the form given in Eq. (17) where $\kappa_i = \mathbf{u}^{\mathrm{T}}\mathbf{K}_i\mathbf{u} \geqslant 0$ and $\mu_i = \mathbf{u}^{\mathrm{T}}\mathbf{M}_i\mathbf{u} \geqslant 0$. Thus,

$$f(\mathbf{b}) = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}} = \frac{\kappa_0 + \sum_{i=1}^{m} \kappa_i b_i}{\mu_0 + \sum_{i=1}^{m} \mu_i b_i}. \tag{24}$$

This quotient does not have any local maximum or minimum where all the b_i are positive. This may be proved by differentiating equation (24) to show that the turning points of f occur when $[\boldsymbol{\kappa}\boldsymbol{\mu}^{\mathrm{T}} - \boldsymbol{\mu}\boldsymbol{\kappa}^{\mathrm{T}}]\mathbf{b} = -(\mu_0\boldsymbol{\kappa} - \kappa_0\boldsymbol{\mu})$, where $\boldsymbol{\kappa} = (\kappa_i)$ and $\boldsymbol{\mu} = (\mu_i)$. Premultiplying by $\boldsymbol{\alpha}^{\mathrm{T}}$, where $\boldsymbol{\alpha}$ is orthogonal to $\boldsymbol{\mu}$, but where $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\kappa} \neq 0$, gives $\boldsymbol{\mu}^{\mathrm{T}}\mathbf{b} = -\mu_0$. Since all the μ coefficients are positive, at least one element of $\mathbf{b}$ must be negative.

Thus,

$$\underset{\mathbf{b}\in\mathbf{b}^I}{\text{extremum}}\left\{\frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}}\right\} = \underset{\mathbf{b}\in\hat{\mathbf{b}}}{\text{extremum}}\left\{\frac{\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u}}\right\}, \tag{25}$$

that is the extremum occur on the vertices of the parameter space. Hence, the extremum values of λ_i all occur at these parameter vertices. □

5. The eigenvalue inclusion principle [22]

To obtain the sharp bounds on the natural frequency of structures, full use will be made of the structure of the mass and stiffness matrices given in Eq. (17). Define the following matrices:

$$\underline{\underline{\mathbf{K}}} = \sum_{i=1}^{m} \underline{b}_i \mathbf{K}_i, \quad \overline{\overline{\mathbf{K}}} = \sum_{i=1}^{m} \bar{b}_i \mathbf{K}_i, \quad \underline{\underline{\mathbf{M}}} = \sum_{i=1}^{m} \underline{b}_i \mathbf{M}_i, \quad \overline{\overline{\mathbf{M}}} = \sum_{i=1}^{m} \bar{b}_i \mathbf{M}_i. \tag{26}$$

Obviously, $\underline{\underline{\mathbf{K}}}, \overline{\overline{\mathbf{K}}}, \underline{\underline{\mathbf{M}}}$ and $\overline{\overline{\mathbf{M}}}$ are real stiffness matrices and mass matrices. Furthermore, it is clear since the $\mathbf{K}_i$ matrices are positive semi-definite, and for any vector $\mathbf{u} \in \mathbb{R}^n$, that

$$\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{K}}}\mathbf{u} \leqslant \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{b})\mathbf{u} \leqslant \mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{K}}}\mathbf{u}, \quad \mathbf{b} \in \mathbf{b}^I. \tag{27}$$

Similarly for the mass matrix,

$$\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{M}}}\mathbf{u} \leqslant \mathbf{u}^{\mathrm{T}}\mathbf{M}(\mathbf{b})\mathbf{u} \leqslant \mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{M}}}\mathbf{u}, \quad \mathbf{b} \in \mathbf{b}^I. \tag{28}$$

Thus,

$$\lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i] = \min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\frac{\sum_{i=1}^{m} b_i^I(\mathbf{u}^{\mathrm{T}}\mathbf{K}_i\mathbf{u})}{\sum_{i=1}^{m} b_i^I(\mathbf{u}^{\mathrm{T}}\mathbf{M}_i\mathbf{u})}\right\} = \min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\frac{\sum_{i=1}^{m}[\underline{b}_i, \bar{b}_i](\mathbf{u}^{\mathrm{T}}\mathbf{K}_i\mathbf{u})}{\sum_{i=1}^{m}[\underline{b}_i, \bar{b}_i](\mathbf{u}^{\mathrm{T}}\mathbf{M}_i\mathbf{u})}\right\}. \tag{29}$$

Using Eqs. (27) and (28), and the properties of interval division,

$$\begin{aligned}\lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i] &= \min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\frac{[\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{K}}}\mathbf{u},\ \mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{K}}}\mathbf{u}]}{[\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{M}}}\mathbf{u},\ \mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{M}}}\mathbf{u}]}\right\} \\ &= \min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\left[\frac{\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{K}}}\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{M}}}\mathbf{u}}, \frac{\mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{K}}}\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{M}}}\mathbf{u}}\right]\right\}.\end{aligned} \tag{30}$$

Hence

$$\lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i] = \left[\min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\frac{\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{K}}}\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{M}}}\mathbf{u}}\right\}, \quad \min_{\mathbf{\Phi}_i\subset\mathbb{R}^n} \max_{\substack{\mathbf{u}\in\mathbf{\Phi}_i\\ \mathbf{u}\neq\mathbf{0}}} \left\{\frac{\mathbf{u}^{\mathrm{T}}\overline{\overline{\mathbf{K}}}\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\underline{\underline{\mathbf{M}}}\mathbf{u}}\right\}\right] \tag{31}$$

and this leads to the following theorem.

Theorem—Eigenvalue Inclusion Principle. *Suppose that a non-negative decomposition of the mass and stiffnesses matrices exists, given by Eq.* (17)*, and where the structural parameter vector* **b** *varys* *inside an interval parameter vector,* $\mathbf{b} \in \mathbf{b}^I$. *Then the ith eigenvalue,* λ_i, *is contained in an interval,* $\lambda_i \in \lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i]$, *where* $\underline{\lambda}_i$ *is an eigenvalue of* $(\overline{\overline{\mathbf{M}}}, \underline{\underline{\mathbf{K}}})$ *and* $\overline{\lambda}_i$ *is an eigenvalue of* $(\underline{\underline{\mathbf{M}}}, \overline{\overline{\mathbf{K}}})$, *where these matrices are defined in Eq.* (26).

The bounds predicted by the parameter vertex solution are tighter than those predicted by the eigenvalue inclusion principle. To prove this, from Eqs. (27) and (28), for the lower eigenvalue bound,

$$\frac{\mathbf{u}^{\mathrm{T}} \underline{\underline{\mathbf{K}}}\, \mathbf{u}}{\mathbf{u}^{\mathrm{T}} \overline{\overline{\mathbf{M}}} \mathbf{u}} \leqslant \frac{\mathbf{u}^{\mathrm{T}} \mathbf{K}(\mathbf{b}) \mathbf{u}}{\mathbf{u}^{\mathrm{T}} \mathbf{M}(\mathbf{b}) \mathbf{u}} \quad \text{for any } \mathbf{b} \in \mathbf{b}^I. \tag{32}$$

Hence the lower bound for the eigenvalue inclusion principle must be lower than that obtained by the parameter vertex solution. A similar situation occurs for the upper bound, showing that that bounds obtained by the parameter vertex solution are tighter. Suppose that the parameter set is disjoint, so that the mass and stiffness are functions of different parameters, and are not a function of the same parameter. In this case the bounds obtained by the eigenvalue inclusion principle occurs at a parameter vector that is at one of the vertices. In this case, the bounds estimated by the eigenvalue inclusion principle and the parameter vertex solution theorem will be identical.

6. Numerical examples

Three examples will be used to demonstrate the procedures outlined on this paper. The first is a discrete mass, spring system, and has the advantage of being easy to implement. The second is a stepped beam, where the cross-sectional area and second moment of area are assumed to vary independently. In this case the stiffness and mass parameters are disjoint and hence the eigenvalue inclusion principle gives the same bounds as the parameter vertex solution. The last example is a truss structure where the parameters are the cross-sectional area of the bar elements. These parameters affect both the mass and the stiffness simultaneously, and thus the eigenvalue inclusion principle and the parameter vertex solution produce different bounds. This will usually be the case in practice when tolerances are given on geometric parameters.

6.1. A spring–mass system

Consider first the five degrees of freedom spring–mass system considered in Refs. [13–15] and shown in Fig. 3. The interval stiffness parameters are

$$k_1^I = [2000, 2020]\,\mathrm{N/m}, \quad k_2^I = [1800, 1850]\,\mathrm{N/m}, \quad k_3^I = [1600, 1630]\,\mathrm{N/m},$$
$$k_4^I = [1400, 1420]\,\mathrm{N/m}, \quad k_5^I = [1200, 1210]\,\mathrm{N/m}$$

and the interval mass parameters are

$$m_1^I = [29, 31]\,\mathrm{kg}, \quad m_2^I = [26, 28]\,\mathrm{kg}, \quad m_3^I = [26, 28]\,\mathrm{kg},$$
$$m_4^I = [24, 26]\,\mathrm{kg}, \quad m_5^I = [17, 19]\,\mathrm{kg}.$$

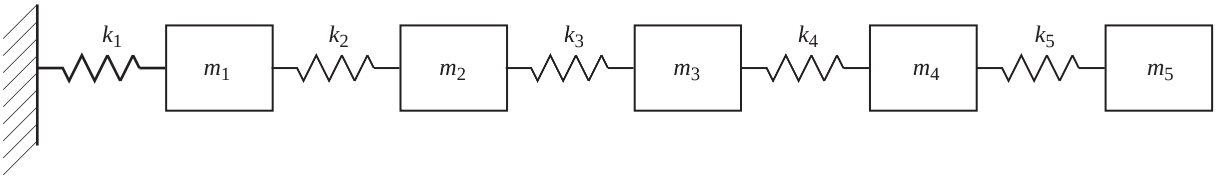

Fig. 3. The spring–mass system with uncertain parameters.

The non-negative decomposition of the global mass and stiffness matrices are

$$\mathbf{M} = \sum_{i=1}^{5} m_i \mathbf{M}_i, \quad m_i \in m_i^I, \quad i = 1, 2, 3, 4, 5,$$

$$\mathbf{K} = \sum_{i=1}^{5} k_i \mathbf{K}_i, \quad k_i \in k_i^I, \quad i = 1, 2, 3, 4, 5,$$

where $\mathbf{M}_i$ and $\mathbf{K}_i$ may be easily derived.

The global stiffness matrix corresponding to the stiffness parameter upper bound vector $\overline{\mathbf{k}} = (2020, 1850, 1630, 1420, 1210)^{\mathrm{T}}$ is

$$\overline{\overline{\mathbf{K}}} = \begin{bmatrix} 3870 & -1850 & & & \\ -1850 & 3480 & -1630 & & \\ & -1630 & 3050 & -1420 & \\ & & -1420 & 2630 & -1210 \\ & & & -1210 & 1210 \end{bmatrix}$$

and the global stiffness matrix corresponding to the stiffness parameter lower bound vector $\underline{\mathbf{k}} = (2000, 1800, 1600, 1400, 1200)^{\mathrm{T}}$ is

$$\underline{\underline{\mathbf{K}}} = \begin{bmatrix} 3800 & -1800 & & & \\ -1800 & 3400 & -1600 & & \\ & -1600 & 3000 & -1400 & \\ & & -1400 & 2600 & -1200 \\ & & & -1200 & 1200 \end{bmatrix}.$$

Similarly, $\overline{\mathbf{m}} = (31, 28, 28, 26, 19)^{\mathrm{T}}$, $\overline{\overline{\mathbf{M}}} = \mathrm{diag}(31, 28, 28, 26, 19)$, $\underline{\mathbf{m}} = (29, 26, 26, 24, 17)^{\mathrm{T}}$ and $\underline{\underline{\mathbf{M}}} = \mathrm{diag}(29, 26, 26, 24, 17)$.

Table 1 summarizes the interval eigenvalues of the spring–mass system obtained by the eigenvalue inclusion principle, and Table 2 lists the interval eigenvalues calculated by the extended Deif's method [13–15]. The proposed eigenvalue inclusion principle yields tighter bounds, namely the lower bounds within the eigenvalue inclusion principle are larger than or equal to those predicted by Deif's method. Likewise, the upper bounds furnished by the eigenvalue inclusion principle are smaller than or equal to those yielded by Deif's approach. This feature clearly demonstrates that the eigenvalue inclusion principle has advantages over Deif's method. Table 3 gives the interval eigenvalues calculated by the interval perturbation method of Qiu et al. [14]. The

Table 1
Interval eigenvalues obtained by the proposed method

	$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\overline{\lambda}_i - \underline{\lambda}_i$
λ_1	5.85810	6.50204	0.64394
λ_2	42.02932	46.30879	4.27947
λ_3	98.85636	108.68899	9.83263
λ_4	158.05142	173.77763	15.72621
λ_5	209.51477	230.08447	20.56970

Table 2
Interval eigenvalues obtained by Deif's method [13]

	$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\overline{\lambda}_i - \underline{\lambda}_i$
λ_1	4.61658	7.83029	3.21371
λ_2	40.64282	47.82004	7.17722
λ_3	98.18002	109.39931	11.21929
λ_4	157.84330	174.00287	16.15957
λ_5	209.51477	230.08447	20.56970

Table 3
Interval eigenvalues obtained by the perturbation method [14]

	$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\overline{\lambda}_i - \underline{\lambda}_i$
λ_1	4.616580	7.830290	3.213710
λ_2	40.753560	47.702460	6.948900
λ_3	98.572270	108.985760	10.413490
λ_4	158.863200	172.894650	14.031450
λ_5	211.504350	227.948750	16.444400

results show that the width of the first three interval eigenvalues of the spring–mass system by the eigenvalue inclusion principle are smaller than those obtained by the interval perturbation method. However, the width of the intervals corresponding to eigenvalues 4 and 5 obtained by the eigenvalue inclusion principle are larger than those predicted by the interval perturbation method. However, it should be emphasized that the bounds obtained by the perturbation approach are only approximate, and that Table 1 gives the exact bounds for this problem.

6.2. A stepped beam

This example considers the stepped beam shown in Fig. 4, where the mass densities and lengths of the three elements are

$$\rho_i = 7800\,\text{kg/m}^3, \quad L_i = 0.4\,\text{m}, \quad i = 1, 2, 3.$$

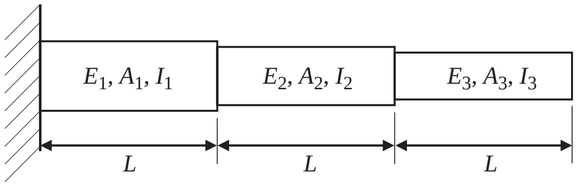

Fig. 4. A three element stepped beam.

Table 4
Interval eigenvalues for the three element beam: Case 1

	Deif's solution theorem		The parameter vertex solution theorem	
	$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\underline{\lambda}_i$	$\overline{\lambda}_i$
λ_1	3.103502E+05	4.196759E+05	3.645921E+05	3.655699E+05
λ_2	7.203591E+06	7.467286E+06	7.327840E+06	7.343228E+06
λ_3	4.804870E+07	4.838791E+07	4.817373E+07	4.826370E+07
λ_4	2.574804E+08	2.586341E+08	2.577919E+08	2.583228E+08
λ_5	8.906338E+08	8.932953E+08	8.910574E+08	8.928784E+08
λ_6	2.736988E+09	2.742580E+09	2.738051E+09	2.741501E+09

Two cases for the stepped beam with uncertain-but-bounded structural parameters will be discussed to compare the parameter vertex solution theorem with Deif's solution theorem.

Case 1: In this case, the cross-sectional areas and the moments of inertia of the elements are deterministic parameters given by

$$A_1 = 1.44 \times 10^{-2}\,\mathrm{m}^2, \quad A_2 = 1.0 \times 10^{-2}\,\mathrm{m}^2, \quad A_3 = 0.64 \times 10^{-2}\,\mathrm{m}^2,$$
$$I_1 = 0.2 \times 10^{-4}\,\mathrm{m}^2, \quad I_2 = 0.1 \times 10^{-4}\,\mathrm{m}^2, \quad I_3 = 0.05 \times 10^{-4}\,\mathrm{m}^2.$$

Young's moduli of the elements are uncertain-but-bounded parameters and their interval values are

$$E_1^I = [199.7, 200.3]\,\mathrm{GN/m}^2, \quad E_2^I = [199.8, 200.2]\,\mathrm{GN/m}^2, \quad E_3^I = [199.9, 200.1]\,\mathrm{GN/m}^2.$$

Table 4 shows the upper and lower bounds of the eigenvalues calculated by Deif's solution theorem and the proposed parameter vertex solution theorem, and shows that the bounds predicted by the parameter vertex solution are much tighter. The model has six degrees of freedom and all six eigenvalues are shown, although, of course, only the lower eigenvalues of a finite element analysis have any physical meaning.

Case 2: In this case, the determinstic parameters are the Young's moduli of elements and their mean values

$$E_i = 200\,\mathrm{GN/m}^2, \quad i = 1, 2, 3.$$

Table 5
Interval eigenvalues for the three element beam: Case 2

	Deif's solution theorem		The parameter vertex solution theorem	
	$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\underline{\lambda}_i$	$\overline{\lambda}_i$
λ_1	5.067125E+04	6.638112E+05	3.612794E+05	3.689557E+05
λ_2	6.636282E+06	8.060068E+06	7.257447E+06	7.415161E+06
λ_3	4.598797E+07	5.071412E+07	4.770923E+07	4.873816E+07
λ_4	2.470625E+08	2.710061E+08	2.553292E+08	2.608389E+08
λ_5	8.289696E+08	9.699690E+08	8.824674E+08	9.016557E+08
λ_6	2.249725E+09	3.613547E+09	2.711208E+09	2.768894E+09

The cross-sectional area and moments of inertia of the elements are taken as uncertain-but-bounded parameters and their interval values are

$$A_1^I = [1.426 \times 10^{-2}, 1.454 \times 10^{-2}]\,\mathrm{m}^2, \qquad A_2^I = [0.99 \times 10^{-2}, 1.01 \times 10^{-2}]\,\mathrm{m}^2,$$
$$A_3^I = [0.634 \times 10^{-2}, 0.646 \times 10^{-2}]\,\mathrm{m}^2, \qquad I_1^I = [0.1998 \times 10^{-4}, 0.2002 \times 10^{-4}]\,\mathrm{m}^4,$$
$$I_2^I = [0.0999 \times 10^{-4}, 0.1001 \times 10^{-4}]\,\mathrm{m}^4, \qquad I_3^I = [0.04995 \times 10^{-4}, 0.05005 \times 10^{-4}]\,\mathrm{m}^4.$$

Table 5 shows the upper and lower bounds of the eigenvalues, which are calculated by Deif's solution theorem and the proposed parameter vertex solution theorem, and shows that the bounds predicted by the parameter vertex solution are much tighter.

6.3. An eight bar truss

The eight bar, pin jointed, truss structure, shown in Fig. 5, will be used to compare the parameter vertex solution theorem and the eigenvalue inclusion principle. The cross-sectional areas of members 1, 2, 3, 4 and 6 are considered to be uncertain-bounded variables, and are taken as $A_i^I = [A^c - \beta A^c,\ A^c + \beta A^c]$, $i = 1, 2, 3, 4, 6$, where $A^c = 2.0 \times 10^{-4}\,\mathrm{m}^2$. β is the uncertainty factor that will be varied to check the eigenvalue bounds with increasing uncertainty. The cross-sectional areas of members 5, 7 and 8 are deterministic and are taken as $A_5 = A_7 = A_8 = 1.0 \times 10^{-4}\,\mathrm{m}^2$. The Young's modulus of the material is $E = 200\,\mathrm{GN/m}^2$. Fig. 6 shows the comparison of the eigenvalue bounds of the first four eigenvalues obtained using the eigenvalue inclusion principle (solid line) and the parameter vertex solution theorem (dashed line) as β varies. The parameter vertex solution theorem yields tighter bounds than those obtained by the eigenvalue inclusion principle. However the computational cost of the parameter vertex solution theorem is high as the solution of 256 eigenproblems is required, as opposed to two eigensolutions for the eigenvalue inclusion principle. Note that because the cross-sectional area of the bars affects both the mass and stiffness matrices, the bounds obtained from the two methods are different. As expected, the eigenvalue bounds increase monotonically with increasing β.

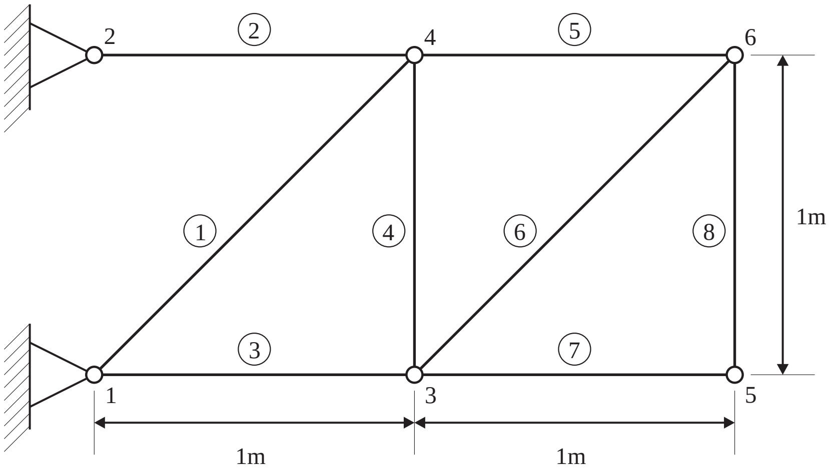

Fig. 5. The eight bar truss example.

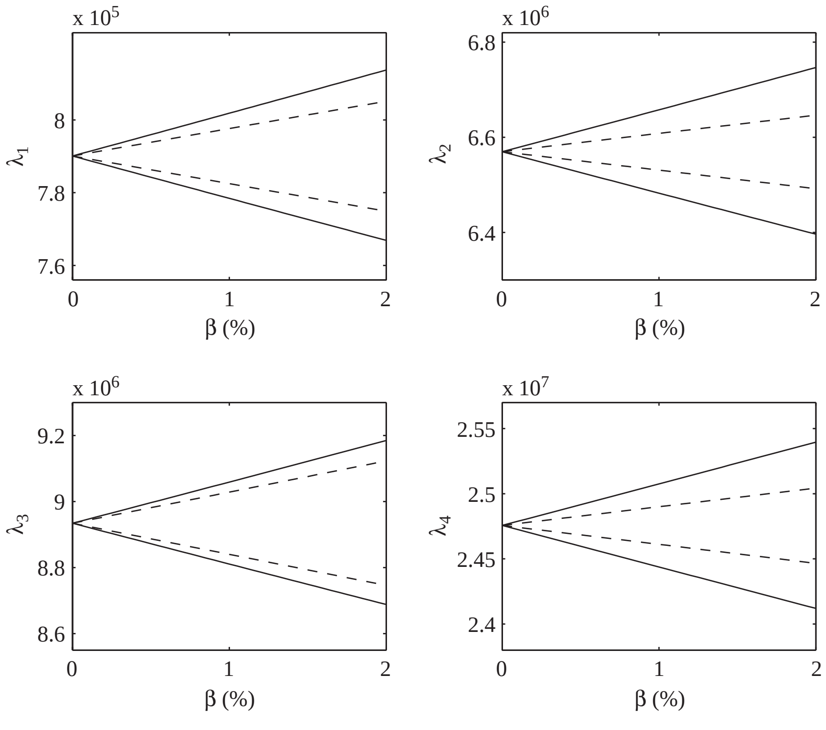

Fig. 6. The bounds on the first four eigenvalues of the eight bar truss (solid line: eigenvalue inclusion principle, dashed line: parameter vertex solution theorem).

7. Conclusion

In this paper the properties of the mass and stiffness matrices in structural engineering, for typical structural parameters have been used to efficiently calculate the bounds on the structure's eigenvalues. The structure in the matrices is defined using the non-negative decomposition of a matrix, and the parameter vertex solution and the eigenvalue inclusion principle are used to determine the lower and upper bounds on the eigenvalues due to uncertain-but-bounded parameters. The effectiveness of the methods was demonstrated by comparison with Deif's solution and the interval perturbation method, using numerical examples. Furthermore, the proposed approaches require minimal computational effort.

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